Part A DEs MT 2007: Sheet 1. (Lectures 1-4)

Material on uniformly convergent sequences and series, needed in this course and covered in Mods is summarised in questions 1 and 2. These should be revision! Questions 3, 4 and 5, are concerned with material I plan to cover in lectures 1-4. KPT

1. Let $[a,b]$ be a closed and bounded interval of the real line and let $\{y_n\}_{n\geq 0}$ be a sequence of real-valued functions, each of which is defined on $[a,b]$. What does it mean to say that **the sequence converges uniformly on** $[a,b]$ **to a limit function** y**?** If each y_n is continuous on $[a,b]$ show that the uniform limit y is continuous on $[a,b]$ and that, when $n \to \infty$,

$$\int_a^b |y_n(x) - y(x)|dx \to 0, \quad \int_a^b y_n(x)dx \to \int_a^b y(x)dx.$$

If $[a,b] = [0,1]$ and $y_n(x) = nxe^{-nx^2}$ show that, for each $x \in [0,1], y_n(x) \to 0$ but $\displaystyle\int_0^1 y_n(x)dx \to \frac{1}{2}$. Thus the convergence must be non-uniform. Show that

$$\max_{0\leq x\leq 1} y_n(x) = \sqrt{\frac{n}{2e}}$$

and sketch the graph of $y_n(x)$ versus x.

2. Let $\displaystyle\sum_{n=0}^{\infty} u_n$ be a series of real-valued functions defined on $[a,b]$. What does it mean to say that **the series converges uniformly on** $[a,b]$? Establish the **Weierstrass M-test** for the uniform convergence of the series, viz. that if there are non-negative numbers $M_n (n \geq 0)$ such that the series $\displaystyle\sum_{n=0}^{\infty} M_n$ converges and $|u_n(x)| \leq M_n$ for every $x \in [a,b]$ and every $n \geq 0$ then the series $\displaystyle\sum_{n=0}^{\infty} u_n$ converges uniformly on $[a,b]$.

Show that the series $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{\cos nx}{1+n^2}$ converges uniformly on $[-\pi, \pi]$.

3. Does the function $F(x,y) = xy^{1/3}$ satisfy a Lipschitz condition on the rectangle $\{(x,y) : |x| \leq h,\ |y| \leq k\}$, where $h > 0$ and $k > 0$?

If $b > 0$ show that the initial-value problem

$$y' = xy^{1/3}, \quad y(0) = b,$$

has a unique solution on any interval $[-c, c]$, with $c > 0$ and find the solution (to get started, find a suitable rectangle and quote Picard - you don't need to prove Picard) .

If $b = 0$ show that for any $c > 0$ there is a solution y which is identically zero on $[-c, c]$ and positive when $|x| > c$.

4. Consider the initial-value problems

$$\begin{aligned} y' &= (1-2x)y, \quad y(0) = 1, \qquad (1) \\ y' &= x^2 + y^2, \qquad y(0) = 0. \qquad (2) \end{aligned}$$

In each case find y_0, y_1, y_2, y_3, where $\{y_n\}_{n\geq 0}$ is the sequence of Picard approximations, and in case (1) show that the sequence converges to the solution for all x. By considering the behaviour of $x^2 + y^2$ on the square $\{(x, y) : |x| \leq \frac{1}{\sqrt{2}},\ |y| \leq \frac{1}{\sqrt{2}}\}$ and appealing to Picard's theorem (Thm. 1.5 of the lectures) show that in case (2) the sequence converges uniformly for $|x| \leq \frac{1}{\sqrt{2}}$.

5. Show (by integrating twice and interchanging the order of integration) that if u is a solution of the initial-value problem

$$u'' = -x^2 u, \qquad u(0) = 1, \qquad u'(0) = 0,$$

then it is a solution of the integral equation

$$u(x) = 1 - \int_0^x (x-s)s^2 u(s)ds$$

Show that if $\{u_n\}_{n\geq 0}$ is the sequence of Picard approximations for the integral equation then the functions u_n are the partial sums of the power series

$$\sum_{n=0}^{\infty} a_n x^{4n}$$

in which $a_0 = 1$ and $a_{n+1} = -\dfrac{a_n}{(4n+3)(4n+4)} \quad (n \geq 0)$.

Show further that if $y = -u'/u$ then y is the solution of the initial-value problem (2) of question 4. Hence write y as the ratio of two power series.